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Testing the Existence of Cobb-Douglas and CES Production Functions in Nigeria

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Abstract

Following the numerous arguments for and against the production functions, this research work tries to find out if the production functions, particularly the Cobb-Douglas and Constant Elasticity of Substitution (CES) production functions holds for Nigeria using an annual time series data from 1970-2012. The Cobb-Douglas and CES production functions which are originally non-linear were linearized and the Ordinary Least Square (OLS) methodology was used. This study also carried out Error Correction Models for each of the production functions in order to know how disequilibrium in output adjusts in the short run for both production functions. It was discovered that both production functions are suitable for the Nigeria economy. The Cobb-Douglas as well as constant elasticity of substitution production function exhibit increasing returns to scale. The results also showed that labour contributes more to output than capital does. Again, for both production functions their error correction models showed that a greater fraction of disequilibrium in output is adjusted in the short run. For the error correction model for Cobb-Douglas it was shown that 87% of disequilibrium in output is adjusted in a year while that for constant elasticity of substitution production function showed 88% of disequilibrium in output is adjusted in a year. The study therefore concludes that the Cobb-Douglas and constant elasticity of substitution production functions are appropriate for the Nigerian economy. The study recommends that both production functions can be used to make forecast in Nigeria.

1.0 Introduction

Production function has been used as an important tool of economic analysis in the tradition of neo-classical economics. It is generally believed that the first economist to algebraically formulate the relationship between input and output was Philip Wicksteed (1984) as:

$$P = f(x_1, x_2, \dots, x_m) \quad (1)$$

But there are some evidences suggesting that Johann Von Thunen first formulated in the 1840's (Humphrey, 1997). It is worth noting that there are two leading concepts of eficency relating to a production system: the technical and allocative efficiency (see Libenstein et al., 1988). According to Shephard (1970), a production function is correctly defined as a relationship between the maximal technically feasible output and the input need to produce that given output. This is because the formulation of production function assumes that the engineering and managerial problems of technical efficiency have been taken care of and solved so that the analysis can now focus on the problem of allocative efficiency.

From the early 1950's to the late 1970's, production function interested many economists. During this period, different specifications relating inputs to output were proposed, well analyzed and used to derive various conclusions. There were single output production functions, multi-output production function and aggregate output production function. Humphrey (1997) gives an outline of historical development of the concept and mathematical formulation of production functions before the enunciation of Cobb-Douglas function in 1928. Paul Douglas, on a sabbatical at Amherst, asked mathematics professor Charles W. Cobb to suggest an equation describing the relationship among the time series on manufacturing output, labour input, and capital input that Douglas had assembled for the period 1889–1922, and this led to their joint paper.

An implicit formulation of production functions dates back to Turgot. In his 1767 observation on a paper by Saint-P´eravy, Turgot discussed how variations in factor proportions affect marginal productivies (Schumpeter, 1954). The logarithmic production function was introduced Malthus (Stigler, 1952) and Bakari in 1959 demonstrated how Ricardo's quadratic production was implicit in his table. Blaug in 1985 showed how Ricardo used his quadratic production function to predict the trend of rent's distributive share as the economy approaches the stationary state.

Johann von Thünen was perhaps the first economist who implicitly formulated the exponential production function as:

$$P = f(A) = A\prod_{1}^{3}(1 - e^{-a_i F_i}) \quad (2)$$

Where F_1, F_2 and F_3 are three inputs labour, capital and fertilizer. a_i are the parameters and P is the agricultural production. In 1969, Lloyd provided a complete account of von Thünen's exponential production functions and their derivation. He was also the first economist to apply the differential calculus to productivity theory and perhaps the first to use Calculus to solve economic optimization problems and interpret marginal productivities essentially as partial derivatives of the production function (Blaug, 1985).

Velupillai (1973) shows out how Wicksell formulated his production function in 1900-1901 that is identical to Cobb-Douglas function. In his 1923 review of Gustaf Akerman's doctoral dissertation Realkapital und Kapitalzins, Wicksell wrote his function as:

$$P = cL^{\alpha}C^{\beta}$$

With the exponents adding up to unity. Works of Turgot, von Thünen, and Wicksell might not have been known to Paul Douglas, but it is surprising to know that before he worked with Cobb, Sidney Wilcox, a research assistant of Douglas, had formulated in 1926 a production function of which the Cobb-Douglas function is only a special case (Samuelson, 1979). Wilcox's production function was, perhaps, ignored by Douglas and till date it has remained in obscurity. In the Cobb-Douglas production function the elasticity of substitution of capital for labour is fixed to unity. The production function formulated by Arrow, Chenery, Minhas and Solow (1961) permitted it to lie between zero and infinity, but to stay fixed at that number along and across the isoquants-irrespective of the size of output or inputs (capital and labour) used in the production process. This function is well known as the Constant Elasticity of Substitution (CES) production function. The CES function takes the form

$$Y = \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{1}{\rho}} \quad (3)$$

It encompasses the Cobb-Douglas, the Leontief and the linear production functions as its special cases. Apart from these production functions, many others were proposed (see Mishra, 2007).

Each of these production functions have been used to reach different conclusion and to predict the amount of inputs necessary to produce a given amount of input. This is the spring board of this research. This research intends to relate inputs to output under the Cobb-Douglas and constant elasticity of substitution (CES) specifications.

Production functions were initially designed with individual firms in mind, but macroeconomists have come to realize that this production function is an essential tool for estimating certain parameters that cannot be directly estimated from national account data. These parameters include partial elaticities of the individual inputs (mostly capital and labour inputs) and the elasticity of substitution between capital and labour.

During the post-Great depression period till the end of the World War II, economists investigated into possibilities of growth with no violent fluctuations. The aggregate production function proved to be very useful in this investigation. The Cobb-Douglas production function was quite cooperative in incorporating technical change introduced in the production system from time to time without changing the basic conclusion on factor shares. The von Neumann growth model (1937) moved from using the Cobb-Douglas production function but retained the practice of aggregation. This line of investigation progressed with the development of linear programming as an optimization method. The activity analysis of Koopmans, the input-output analysis of Leontief, the aggregate linear production function of Georgescu-Roegen (1951), separation theorems and generalization of von Neumann's model by Gale (1956), careful proofs given by Nikaido (1968) all strengthened the foothold of aggregate production function in economic analysis.

Notwithstanding this progress, the 1950's did not accept the aggregate production function wholeheartedly. Mrs, Joan Robinson (1953) viewed aggregate production function with a remark '… the production function has been a powerful instrument of miseducation. The student of economic theory is taught to write $Q = f(L, K)$ where *L* is a quantity of labour, *K* a quantity of capital and *Q* a rate of output of commodities. He is instructed to assume all workers alike, and to measure *L* in man-hours of labour; he is told something about the index-number problem in choosing a unit of output; and then he is hurried on to the next question, in the hope that he will forget to ask in what unit *K* is measured. Before he ever does ask, he has become a professor, and so sloppy habits of thought are handed on from one generation to the next.'

A controversy began, especially regarding the measurement of capital, in which Piero Sraffa, Joan Robinson, Luigi Pasinetti and Pierangelo Garegnani (among others) argued against the use of aggregate production function, and Paul Samuelson, Robert Solow, Frank Hahn and Christopher Bliss (among others) argued in favour of using aggregate production function for explaining relative factor shares. As argued by the first group, it is impossible to conceive of an abstract quantity of capital which is independent of the rates of interest and wages. However, this independence is a precondition of constructing an isoquant (or production function). The iso-quant cannot be constructed and its slope measured unless the prices are known beforehand, but the protagonists of aggregate production function use the slope of the isoquant to determine relative factor prices. This is begging the question.

In spite of all the arguments against the production function, the supporters of the aggregate production function did not relent. Once Paul Samuelson (1966) wrote: 'Until the laws of thermodynamics are repealed, I shall continue to relate outputs to inputs - i.e. to believe in production functions. Unless factors cease to have their rewards to be determined by bidding in quasi-competitive markets, I shall adhere to (generalized) neoclassical approximations in explaining their market remunerations.' One does not seem to understand as to how the validity of the laws of thermodynamics hold. Relating outputs to inputs entails validity of the proposition that the aggregate of functions would be the function of aggregates, and if there is some particular type of function that has this property then that particular function is the correct function describing the production technology of any economy and the relative factor shares.

Solow (1957) made a remarkable empirical study to show how the aggregate production function fits to the U.S. data for 1909-1949, which confirm neutral technical change, shift in production function, and therefore validated the artifact of aggregate production function as a powerful tool of analysis.

These arguments have led to this research. This study is interested to find out if aggregate production functions particularly, the Cobb-Douglas and CES production functions really exist? Put in another way, the study, intends to find if the Cobb-Douglas or/and CES production function can be used to reach conclusions regarding inputs and outputs in the Nigerian economy or not.

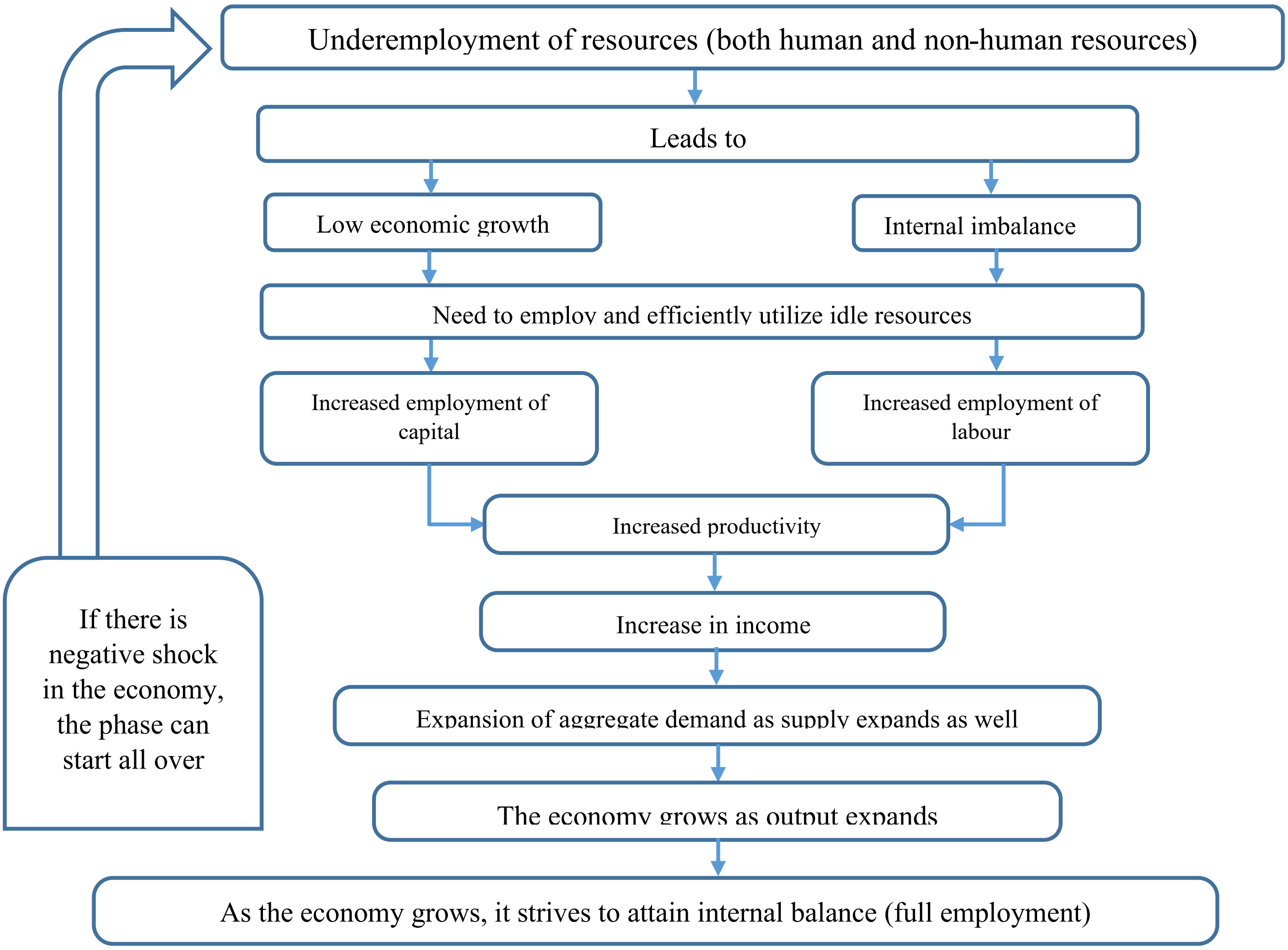

Figure1: A Flowchart Showing the Relationships among the Variables of Interest; Capital, Labour and Output

In this study gross fixed capital formation is used as a proxy for capital input. According to National Bureau of Statistics (2011), Gross fixed capital formation is measured by the total value of a producer's acquisitions, less (minus) disposals of fixed assets during the accounting period plus certain additions to the value of non-produced assets (such as sub-soil assets or major improvement in the quantity, quality or productivity of land) realized by the productive activity of institutional units. Labour input is the number of hours worked or the number of employed people. It is used in this study as the number of people who fall within the working age bracket and are willing and able to work and also able to find paid employment. It is the proportion or fraction of the labour force that are employed. Economic output is the total output an economy produces in a particular period say one year. In this study we use nominal gross domestic product (NGDP). Nominal gross domestic product is the monetary value of all final goods and services produced in a given year at current prices. It is worth noting that returns to scale is the changes in production that occur when the factors of production are proportionally changed. It is quantitative change in outputs resulting from a proportionate change in all inputs.

1.2 Theoretical Literature

Adam's theory of economic growth is gotten from his well reputed book, '*Wealth of Nations*' which was written in 1776. The model deals with capitalist economies and their process of economic growth. It is concerned with the process that enabled the developed and rich nations to grow. In this model, Adam Smith discussed the following; production function, natural resources, institutions, labour force and capital accumulation. In his production function, he includes land and capital in addition to labour. Adam's production function shows that the production of the economy is dependent on labour, land and capital.

$$Y = f(L, K, N) \quad (4)$$

Where $Y = National\ product$, $L = Labour$, $K = Capital$, and $N = Natural\ resources$. This production function is not subject to diminishing returns but increasing returns. He argues that as production rises, both internal and external economies of scale are attained. As a result, the market will be extended and real costs of production will decrease. As a result of economies of scale, there is room for division of labour as well as improvement of machinery. Varying degrees of division of labour yields varying labour productivity. Division of labour is constrained by the extent of the market and the size of the market is affected by institutional framework and amount of capital. Labour productivity and size of the market to him, are influenced by regulation of local and foreign trade. That is why he advocated for domestic and international division of labour and specialization. He also said that labour productivity (MP_L), land productivity (MP_N) are influenced by stock of capital (K) and institutional framework (U).

According to him, change in institutional framework as time goes by is determined exogenously. As a result, the variable institutional framework (U) is considered fixed. Also, for him land is fixed and the growth rate of an economy is dependent on changes in capital accumulation and labour force with respect to time. Smith further emphasized that supply of labour is related to population and in the long run population is affected by wages paid for labour. If labour are given more actual wages above the subsistence wages, then marriages will take place resulting to increase in the population. The opposite would be the case if labour are given actual wages which are below the subsistence wages. If actual wages equal subsistence wages, population will remain unchanged. Demand for labour is determined by wage fund. For him, there is a particular amount of wage fund in the economy. That amount of wage fund determines labour demand. In other words, demand for labour is dependent on changes in capital and changes in income. In line with this, the growth of the labour force depends on capital growth and income growth. Capital accumulation depends on investment and investment depends on savings. As he says 'capitals are increased by parsimony and diminished by misconduct.' Savings is determined by the consideration of private profit. Profit stimulates the desire to save and invest. Accumulation of capital continues as long as the rate of profit exceeds the amount of compensation for the risk from investment. As capital stock grows, the rate of profit declines. The rate of profit is also determined by institutional framework implying that the degree of commerce, control over monopoly or competition and the restrictions over international trade influence the rate of profit. Also, capital accumulation is affected by interest rate. As interest rate falls, people will come to terms that they can no longer live on property income. They now turn to business. This will lead to increased capital accumulation. In a growing economy, national product increases as accumulation of capital rises but with increase in capital, the rate of profit falls. In summary, in Smith's model capital accumulation will increase in an economy that is growing. This will result to increase in the level of output of the economy.

Ricardo's model of economic growth (1812) includes the production function, natural and human resources, capital accumulation and pattern of development. Ricardo's production function is given as

$$Y = f(L, N, K) \tag{5}$$

Where $Y = output$, $L = labour$, $K = capital$, $N = land$, This production function is subject to diminishing marginal productivity. This implies that as more of K, L, and N are employed, their respective marginal productivity falls. However, Ricardo thinks that diminishing marginal productivity can be addressed through technological progress in an economy. The production function becomes

$$Y = f(L, N, K, S) \tag{6}$$

Where $S = Technology$. Differentiating totally,

$$dY = \frac{\partial Y}{\partial L}.dL + \frac{\partial Y}{\partial N}.dN + \frac{\partial Y}{\partial K}.dK + \frac{\partial Y}{\partial S}.dS \tag{7}$$

Dividing through by dt

$$\frac{dY}{dt} = \frac{\partial Y}{\partial L}.\frac{dL}{dt} + \frac{\partial Y}{\partial N}.\frac{dN}{dt} + \frac{\partial Y}{\partial K}.\frac{dK}{dt} + \frac{\partial Y}{\partial S}.\frac{dS}{dt} \tag{8}$$

This implies that Ricardo's model depends on change in capital accumulation with respect to time $\left[\frac{\partial K}{\partial t}\right]$, change in labour with respect to time $\left[\frac{\partial L}{\partial t}\right]$, change in land with respect to time $\left[\frac{\partial N}{\partial t}\right]$, and change in technology with respect to time $\left[\frac{\partial S}{\partial t}\right]$. According to Ricardo, land encompasses the original and indestructible powers of the soil. As a result of this, he considers land to be fixed in supply because it is a gift of nature. According to him, there are two kinds of wage rate: natural wage rate and market wage rate. Natural wage rate is similar to subsistence wage rate in Adams Smith's model. If market wage rates exceed the natural wage rate, the population will grow and vice versa. Natural wage rate is determined by productivity and socio cultural environment. Changes in market wage rate are caused by the demand and supply of labour. When labour demand exceeds supply, market wage rate rises and vice versa. Market wage rate remains constant when the demand and supply of labour are equal. Labour demand is influenced by stock of capital in the economy. In the long run, labour demand equals labour supply. This implies that long run growth of labour force depends on capital and there is a proportional relationship between the two variables. For Ricardo, capital includes both fixed and circulating capital. Fixed capital is that part of a country's wealth that is employed in production and it includes food, tools, machinery, raw materials and clothing. Circulating capital consists of wage fund. It grows in constant proportion to fixed capital as long as there are no technical changes taking place in the economy. Ricardo says that capital is accumulated as a result of increase in revenue and decline in consumption through increased savings. Rate of capital accumulation is determined by the ability to save and the will to save. The ability to save is dependent on the surplus over the total product necessary to maintain to labour at subsistence level. Higher surplus leads to higher savings. The will to save is dependent on the rate of profit. He said 'while the profits of stocks are higher, men will have motive to accumulate. If the rate of profit falls, men can move towards increased consumption.' Rate of profit rises and falls depending on subsistence wages. Capital accumulation plays an important role in economic development. As a result the growth of an economy is dependent on the initial amounts of accumulated capital, labour, and land, structural parameters like institutional framework and technology. Institutional framework and technology are determined exogenously.

$$Y = f[K_o, L_o, N_o, a \dots \dots aN; U(t); S(t)] \quad (9)$$

Where $K_o = initial\ of\ capital$, $L_o = initial\ amount\ of\ labour$, $N_o = initial\ amount\ of\ land$, $U(t) = institutional\ framework$, and $S(t) = technology$. Ricardo's model is based on diminishing returns and Malthusian theory of population. It is based on what happens in under developed countries. Analyzing under developed countries, it can be seen that population grows more than food grows. The use of enhanced technology on land is limited. As a result, the existence of diminishing returns. In the case of under developed countries, rent increases because supply does not match demand and as a result of abundant labour, wages remain low leading to low income, low savings, high lending rate, and eventually, low investment.

Schumpeter's model of economic growth (1912) revolves round inventions and innovations. It is explained with the following; process of production, dynamic analysis of the economy, trends of growth, and demise of capitalism. The Schumpeter's production function is given as

$$Y = f(L, K, N, S, U) \quad (11)$$

Where $Y = Output\ of\ the\ economy$, $L = Labour$, $K = Produced\ means\ of\ production$, $N = Natural\ resources$, $S = Technology$, and $U = Social\ set\ up\ or\ Social\ organization$. According to the model, production of the economy is determined by the rate of change of productive forces (labour, natural resources, and produced means of production), the rate of change of technology, and the rate of change of social organization. Under the dynamic analysis of the economy, there are two effects which are the growth components ant the evolution components. The growth components have to do with the effects of change in factors of production like labour, natural resources and produced means of production. The evolution components have to do with the effects in technological and social changes. For the growth components, land is fixed. As a result of this change in land $dN/dt = 0$. The other two variables are change in labour dL/dt and change in produced means of production dK/dt. Regarding change in labour, he says population is exogenously determined. For produced means of production or capital goods, Schumpeter says that their change is dependent on savings and savings depends on rate of profit. For the evolution components, regarding the institutional and social changes, he says that it has to do with factors such as social, psychological, political, and technical atmosphere of a country. Schumpeter says that economic development the result of technical change that is not continuous. For him, the process of economic development can be initiated with five events which are; introduction of some new goods, discovery of some new market, discovery of some new source of supply, introduction of some new technique of production, change in the structure and organization of some industry. All these kind of changes lead to changes in the absorption of factors of production.

For Schumpeter, capitalist economies have properties of cyclical fluctuations. This implies that they experience both booms and recessions. The entrepreneurs that invent new techniques of production or some new product, when it is introduced in the market, make heavy profits. After some time, other firms adopt that technique of production or produce that very product. This leads to increase in supply, then fall in price of the product. This leads to fall in revenue and eventually fall in profit thereby creating depression in the economy. Like Karl Marx, Schumpeter is of the view that capitalism will eventually die off and be replaced by socialism. Schumpeter's model is concerned with specific economic and social structure that prevailed in Europe in the 18th and 19th century. The model attaches economic development with just inventions and innovations.

The Solow growth model (1993) consists of variables that is, economic quantities that are measureable. It also consist of behavioral relationships which are relationships that describe how people make economic decisions. That is to say how people decide what to do when given their opportunities and opportunity costs. It is also made up of equilibrium conditions. These are conditions that tell us when the economy is in a position of balance, when the variables we are focusing on are 'stable' – that is, when the variables are changing in simple and predictable ways. Solow began with a production function of the Cobb-Douglas type

$$Y = AK^{a}L^{b} \quad (12)$$

Where $Y = Output$, $A = Multifactor\ productivity$, $K = Capital$, $L = Labour$, $0 > a, b < 1$, $a + b = 1$ Indicating constant returns to scale. Solow stated that any increase in output Y, could be as a result of the following;

1. Increase in labour, L. But due to diminishing returns to scale, increase in labour, implies decline in output per worker or labour productivity (Y/L).
2. Increase in capital, K. Increase in the stock of capital would increase both output and labour productivity.
3. Increase in multifactor productivity (A). This would also lead to increase labour productivity.

In the case of the Solow growth model, the key variable is labor productivity. Labour productivity is defined as output per worker. That is how much the average worker in the economy is able to produce. Output per worker is calculated by simply taking the economy's level of real gross domestic product (RGDP) or output Y, and dividing it by the economy's labor force L. This quantity, output per worker, $Y/_{L}$, is our best simple proxy for the standard of living and level of prosperity of the economy.

To concentrate on what happens to labour productivity, Solow made it the subject of the formula and rewrote equation (13) as

$$Y/L = AK^{a}L^{b-1} = AK^{a}/L^{1-b} \quad (13)$$

Since it was assumed that $a + b = 1$ then $a = 1 - b$. Rewriting equation (14)

$$Y/L = AK^{a}/L^{a} = A(K/L)^{a} \quad (14)$$

Let $Y/L = y$ and let $K/L = k$ we have

$$y = Ak^{a} \quad (15)$$

This is the key formula we will work with. We will examine how the model works when growth comes through capital accumulation, and how it works when growth is due to innovation. In addition to the production function, we need the saving function and the equilibrium condition. Savings function provides information on how people in an economy save output of a given income while the equilibrium condition has to do with a state of balance. If the savings function and equilibrium conditions are known, we can predict the productivity of labour, y. In every economic model, economists analyze the model by looking for equilibrium: a point of balance. Economists look for equilibrium for a simple reason: it is either an economy is at its equilibrium position, or it is moving to an equilibrium position. In economic growth, what economists look for is an equilibrium in which economy's capital stock per worker, its level of real gross domestic product (RGDP) per worker or labour productivity, and its efficiency of labor are all three growing at the exact same proportional rate. Other models may include Harrod-Domar model, J.E. Meade's model, etc.

1.3 Empirical Literature

Abidemi (2010) using ordinary least square (OLS) method examined the productivity of the banking sector in Nigeria from 1960-2008 by estimating two major production functions; the Cobb Douglas production function and the constant elasticity of substitution production function. Form the Cobb Douglas production function, the sum of the substitution parameters were greater than one showing that the production function of the banking sector in Nigeria exhibits increasing returns to scale. The substitution parameters of the constant elasticity of substitution production function were also positive. The study supports economic theory on the specification of both Cobb Douglas and the constant elasticity of substitution production function. Effiong and Umoh (2010) used the Cobb Douglas production function based on stochastic profit frontier and tried to estimate the profit efficiency and the relevant indicies determining efficiency levels for egg-laying industry in Akwa-Ibom state. Their empirical results showed that variable inputs such as price of feeds, price of drugs and medication were statistically significant ($p<0.05$) thus indicating that profit decreased with increase in input prices while fixed inputs such as capital inputs and farm size were statistically significant and had the right sign a-priori indicating that profit increased with increase in the level of its utilization. Adetunji, Ibraheem and Ademuyiwa (2012) using the F-test approach restricted least squares (RLS) examined the Nigerian economy from 1990-2009 to find out if the linear restriction of Cobb- Douglas production ($\beta + \alpha = 1$) is significant to Nigeria. They discovered that the economy of Nigeria is characterized by constant returns to scale over the period that was sampled and using the restricted least square as stipulated by Cobb-Douglas function may not be misleading.

Felipe and McCombie (2001) in their work outlines the Phelps Brown critique and its extensions that at the aggregate level only value data can be used to estimate production function, this means that the estimated parameters of the production function are merely capturing an underlying accounting identity. They concluded that the theoretical basis of the aggregate production function is problematic. Iain (2002) re-examined the original time series data which was used by Professor Douglas and other associated researchers to establish the existence of an aggregate production function. Using various statistical methods, only New South Wales data and that of New Zealand to a lesser extent produced results that support the assertion of Douglas who claimed that the data provide deductive support for the laws of production. Winford and Chris (2003) examined whether nonlinearities in the aggregate production function could explain parameter heterogeneity in the Solow growth regression. They first questioned the empirical relevance of the Cobb Douglas aggregate production specification in cross country linear regressions. They discovered that both in the basic and the extended regression models that the more general CES specification with elasticity of substitution greater than unity is accepted over the Cobb Douglas specification. They found that the CES specification better fits cross-country variation than the Cobb Douglas specification.

Pol (2004) using the US time series data from the period of 1948-1998 tried to verify if the U.S aggregate production function is Cobb Douglas. He first adopted Berndt's specification which assumes technological change is Hicks neutral and got elasticities of substitution that are not significantly different from one. He also modified the econometric specification to allow for biased technical change and obtained significantly lower estimates of the elasticity of substitution. He concluded that the U.S. economy is not well described by a Cobb-Douglas aggregate production function. Ayoe (2004) performed a test of constant elasticity of substitution structure of the n-input translog function for the fleet of Danish trawlers operating in the North Sea from 1987-1999. The translog function was fitted to the monthly landed catch value of the fleet, using fishing time, fishing power, labour, and stock as inputs. He concluded that it is not an approximation to the CES function in the given case. But, CES is present for the fleet, as the appropriate production form is believed to be Cobb-Douglas, having elasticity of substitution equal to unity. Giannis, Theodore, and Chris (2004) examined a one-sector growth model with a variable elasticity of substitution production function. They showed that the model can exhibit unbounded endogenous growth despite the absence of exogenous technical change and the presence of non-reproducible factors such as labour. They also used a panel of eighty-two countries over a period of twent- eight years to an aggregate production function. The empirical estimates of the elasticity of substitution support the possibility of unbounded endogenous growth.

Dana and Jaromír (2007) using data for the period of 1995-2005, tested whether the application of Cobb Douglas production function in Czech economy is unreliable. This is as a result of the fact that the Cobb Douglas production function is often used to analyze the supply side performance of an economy and it assumes a constant share of labour in output that may be too restrictive for a converging country. They applied a more general production and allowed labour share to develop according to empirical data. There was no significant difference between the use of Cobb Douglas and a more general production function in calculating the supply side of Czech economy. Scott (2007) argues that the logical and empirical problems with the Cobb-Douglas were well-known by the most advanced minds of mainstream economics. This calls into question the rationale for its continued use as an empirical corroboration of marginal productivity theory. Eric (2008) assessed the Cobb

Douglas and constant elasticity of substitution production functions to find out which is most appropriate for Congressional Budget Offices (CBO's) macroeconomic forecast. He discovered that Cobb Douglas has seemingly good empirical fit across many data but also fits well in cases where some of its basic assumptions are violated. The constant elasticity of substitution production function has less restrictive assumptions about the way capital and labour interact in production. Unfortunately, econometric estimate of its elasticity parameter have produced inconsistent results. For forecasting purposes, there may not be any strong reason to prefer one over the other but for analysis of policies affecting return of factor, Cobb Douglas production function may be restrictive.
Devesh (2010) in his work provided evidence from U.S manufacturing plants against Cobb Douglas production function and presented an alternative production function. The Cobb Douglas production function has the implication of a constant cost share of capital and strong co movement in labor productivity and capital productivity. But the industries Devesh captured exhibit differences in cost shares of capital over time. A constant elasticity of substitution production function with labour augmenting differences and an elasticity of substitution between labor and capital less than one can account for these facts. He strongly rejected Cobb Douglas and concluded that capital and labor are complements. Sagar, Eric, and Mikael (2013) tried to analyze the statistical relationship between output and inputs of labour and capital in the Belgain labour market. Using OLS to estimate the Cobb-Douglas production they found out that there is a strong relationship between input goods, capital and labour and the output in the Belgian market.
Existing literatures have made efforts in modelling microeconomic agents' productivity using either the cobb-Douglas production function or the Constant Elasticity Substitutes (CES) production function. None of these papers have modelled the macroeconomic productivity level of Nigeria using either of the aforementioned production functions and secondly, this work did not only model the output level of Nigeria using both models but tested the appropriate model between the production functions in modelling the output level of Nigeria.

2.0 Methodology

A production function is a tool that shows the maximum output that can be produced with given levels of inputs using a given technology.
It can be expressed functionally as

$$Q = f[X] \qquad (16)$$

Where X is a vector of factor inputs

$$[X_1, X_2, X_3, \dots, X_n] \qquad (17)$$

And Q is the maximum outputs that can be producing with the given inputs. In microeconomics, all factor inputs are emerged into two broad categories; K= fixed inputs, and L= variable inputs. There are numerous types of production function that can be obtained but the production functions which will be considered in this work are Cobb-Douglas production function and constant elasticity of substitution (CES) production function. The function form of Cobb-Douglas production function is widely used to represent the relationship between output and inputs. Cobb-Douglas production function is of the form

$$Q = AL^{\alpha}K^{\beta} \qquad (18)$$

Where Q = Total output produced (the monetary value of all goods and services produced in a year) A = Total factor productivity, L = Labour input (the total number of workers employed), K = Capital inputs (the monetary value of all machinery, equipment, and buildings), α and β are the respective output elasticities of labour and capital. The values of α and β constant and are determined by the available technology. The returns to scale is the sum of α and β. The sum of α and β also gives the degree of homogeneity of the Cobb Douglas production function. For a Cobb Douglas production function, when:

- $\alpha + \beta < 1$, we have decreasing returns to scale. This means that if the inputs capital and labour are doubled, output will decrease more than proportional to the increase in output. In this case the production function is homogenous of degree less than one.
- $\alpha + \beta = 1$, we have constant returns to scale. Meaning that if inputs are doubled, then output will double as well. Here the production function is homogenous of degree one.
- $\alpha + \beta > 1$, we have increasing returns to scale. What this means is that if inputs are doubled, outputs will more than double. The production is homogenous of degree greater than one.

For the Cobb Douglas production function $\alpha + \beta$ is assumed to be unity. The CES production function is a function that describes production, usually at a macroeconomic level, with two inputs, capital and labour. As defined by Arrow, Chenery, Minhas, and Solow (1961), it is given by:

$$Y = \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{\frac{-1}{\rho}} \quad (19)$$

$$\rho = \frac{1}{\sigma} - 1 \quad (20)$$

$$\sigma = \frac{1}{1+\rho} \quad (21)$$

Where K is a measure of capital input, L is a measure of labour input, $\delta, \rho, and\ \gamma\ are\ constant$. Originally, the CES function of Arrow et al. (1961) could only model constant returns to scale, but later Kmenta (1967) added the parameter v, which allows for decreasing or increasing returns to scale if $v < 1\ or\ v > 1$, respectively. The formal specification of the CES production function is given by

$$Y = \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{v}{\rho}} \quad (22)$$

Where Y is the output quantity, K and L are labour inputs, and $\delta, \rho, \gamma\ and\ v\ are\ constants$. The parameter $\gamma \in= [0,1)$, determines productivity. It is the efficiency parameter and it plays the same role as the coefficient A in the Cobb-Douglas function; it serves as an indicator of the state of technology. The parameter $\delta \in= [0,1]$ determines the optimal distribution of the inputs. It is the distribution parameter like α in the Cobb-Douglas function. The parameter $\rho \in [-1,0) \cup (0,\infty)$ is the substitution parameter. It has no counterpart in in the Cobb-Douglas function. It helps determine the value of the (constant) elasticity of substitution, which is given by $\sigma = \frac{1}{1+\rho}$, and $v \in= [0,\infty)$ is equal to elasticity of scale. $0 \leq \gamma < 1$, $0 \leq \delta \leq 1$, $\rho \neq 0$, $-1 \leq \rho < \infty$, and $0 \leq v < \infty$

$$NGDP_t = \beta GFCF_t^{\alpha_1} LEMP_t^{\alpha_2} \quad (23)$$

$$NGDP_t = \gamma[\delta GFCF_t^{-\rho} + (1-\delta)LEMP_t^{-\rho}]^{-\frac{v}{\rho}} \quad (24)$$

, $lnGFCF$ =$lnNGDP$ = Logarithm of Nominal Gross Domestic Product Logarithm of Gross Fixed Capital Formation , $lnLEMP$ = Logarithm of Labour Employed , $(lnGFCF_t - lnLEMP_t)^2$ = Square of the Difference of $lnGFCF$ and $lnLEMP$ (See appendix a1 for the linearization of the models).

2.1 Results And Interpretation

Table1:
Dickey Fuller Test Result (Drift and Trend)

Variables	DF Statistic (DF_{Cal})	DF critical value at 5%	P-value	Decision	Conclusion
$lnNGDP$	-10.90541	-3.523623	0.0000	Reject H_0	Stationary
$lnGFCF$	-4.806173	-3.523623	0.0020	Reject H_0	Stationary
$lnLEMP$	-8.554662	-3.523623	0.0000	Reject H_0	Stationary
Z	-4.009974	-3.523623	0.0160	Reject H_0	Stationary

$$z = (lnGFCF - lnLEMP)^2$$

The results in table 4.2 shows that at level form, $DF_{Cal} < DF_{Tab}$ for all the variables. This implies that the variables are stationary at first difference when the dickey fuller test is done including no drift and trend, only drift, both the drift and trend.

Table 2:
Ordinary Least Square Regression Result for Cobb-Douglas Production Function

Dependent variable $lnNGDP$					
Variables	Coefficient	Standard Error	T-Value	P-Value	95% confidence interval
$lnGFCF$	0.6812225	0.0952633	7.15	0.000	0.4886881 0.8737569
$lnLEMP$	3.476717	0.6003758	5.79	0.000	2.263313 4.531227
$Constant$	-54.1186	9.284369	-5.83	0.000	-72.88301 -35.35419
$R^2 = 0.9765$ $\bar{R}^2 = 0.9753$	Durbin-Watson d-statistic(3, 43) = 1.738876			F(2, 40) = 829.94 Prob> F = 0.0000	

Table 3:
Ordinary Least Square Regression Result for C.E.S Production Function

Dependent variable $lnNGDP$					
Variables	Coefficient	Standard Error	T-Value	P-Value	95% confidence interval
$lnGFCF$	0.5602974	0.2444533	2.29	0.027	0.658439 1.054751
$lnLEMP$	3.442377	0.6091337	5.65	0.000	2.210288 4.674466
$(lnGFCF - lnLEMP)^2$	-0.0133799	0.024869	-0.54	0.594	-0.0636823 0.0369225

$Constant$	-51.65402	10.428	-4.95	0.000	-72.74663 - 30.5614
$R^2 = 0.9766$ $\bar{R}^2 = 0.9748$	Durbin-Watson d-statistic(4, 43) = 1.737092			F(3, 39) = 543.56 Prob> F = 0.0000	

Table 4: Cobb-Douglas Production Function (Model 1a)

Parameters	Expected outcome	Actual outcome	Status
α_1	$\alpha_1 > 0$	0.6812225	It conforms to a priori
α_2	$\alpha_2 > 0$	3.476717	It conforms to a priori

Table 5: C.E.S Production Function (Model 1b)

Parameters	Expected outcome	Actual outcome	Status
γ	$0 \le \gamma < 1$	$3.689301478 \times 10^{-23}$	It conforms to a priori
δ	$0 \le \delta \le 1$	0.139980758864623	It conforms to a priori
ρ	$\rho \ne 0, \quad -1 \le \rho < \infty$	-0.0555336333080194	It conforms to a priori
v	$0 \le v < \infty$	4.0026744	It conforms to a priori

(See appendix a2). The results above show that there is exist a positive relation between output and inputs. As inputs increases, output also increases. In the evaluation based on econometric criteria, we see that the parameters conform to a priori expectations. From the above, it is also seen that for the period of 1970-2012 in the Nigerian economy, a percentage increase in capital inputs (GFCF) holding labour inputs (LEMP) constant, led to an average of 0.68 percentage increase in output (NGDP). Also, holding capital inputs (GFCF) constant, a percentage increase in labour inputs (LEMP) led to an average of 3.48 percentage increase in output (NGDP). It is also seen that the sum the partial elasticities is greater than one. Hence, the Nigerian economy is characterized by a Cobb-Douglas production function which exhibits increasing returns to scale. From the result of the error correction model (E.C.M) for the Cobb-Douglas production function, it was found that the coefficient of the lagged residuals is-0.8680508. This suggest that output (NGDP) adjust to inputs (GFCF and LEMP) with a lag; 87% of the discrepancy between long-term and short-term NGDP is corrected within a year. Implying that about 87% of the disequilibrium in the past year's shock adjusts back to the long run equilibrium in the present year. Taking the reciprocal of -0.8680508 $({}^{1}/_{-0.8680508} = 1.15)$ in absolute terms, it is seen that it takes a year, a month and 24days for equilibrium to be completely restored. From the result of the error correction model (E.C.M) for the Constant Elasticity of Substitution production function, it was found that the coefficient of the lagged residuals is-0.8765007. This suggest that about 88% of the disequilibrium in the past year's shock adjusts back to the long run equilibrium in the present year. Taking the reciprocal of -0.8765007 $({}^{1}/_{-0.8765007} = 1.14)$ in absolute terms, it is seen that it takes a year, a months and 20days for equilibrium to be completely restored.

3. Conclusion

In this study, the Cobb-Douglas and constant elasticity of substitution aggregate production functions were estimated using ordinary least square (OLS) methodology. The Cobb-Douglas and constant elasticity of substitution production functions are nonlinear in their original form. These functions were linearized and OLS estimation technique applied. From the OLS estimation, the parameters in the two production functions conformed to a priori expectations. This tells that both production functions can be used to make forecast in the Nigerian economy. For each of the models used to estimate the Cobb-Douglas and C.E.S production functions, an error correction model was estimated for each of them. It was discovered that for each of the production functions, disequilibrium in output can be adjusted in both production functions at almost the same rate. The researcher discovered that the parameters in the Cobb-Douglas and constant elasticity of substitution all conformed to a priori expectation. This is in line with the findings of Abidemi (2010) although it was just for the banking sector but it contradicts the finding of Adetunji, Ibraheem and Ademuyiwa (2012) which showed that the Cobb-Douglas production function for Nigeria exhibits constant returns to scale. Also, in the Cobb-Douglas as well as the constant elasticity of substitution production function, it was discovered that output adjusts fast to long run disequilibrium. It was also discovered that labour inputs contribute more to output when compared to capital inputs. This finding is also in line with the findings of Abidemi (2010) in his work; capital-labour substitution and banking performance. This could be as a result of over working the few machines available which could lead to their breakdown.

The researcher thus recommends that

i. Since the study has shown that labour inputs contribute more to output it means that it is rational to employ more of labour inputs in production. However, the rate of unemployment of labour in Nigeria is very high. According to CBN (2012), unemployment rate was 27.07 percent. Thus, it is imperative to reduce this number by providing more jobs for the teeming population.
ii. More capital should also be employed. Capital should be employed to the point that the available ones, say machines are not over employed. If machines are allowed to produce not more than their potential, then capital could contribute more than it used to contribute because it would not breakdown or crash before it exceeds it useful life span.
iii. Furthermore, government in trying to fulfil its aim of welfare maximization, should employ more labour by creating employment opportunities as this will reduce the rate of unemployment of labour which is on the increase as well as reduce social vices usually caused the unemployed.

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APPENDIX A1
LINEARIZATION OF THE COBB-DOUGLAS AND CONSTANT ELASTICITY OF SUBSTITUTION PRODUCTION FUNCTION

Cobb-Douglas Specification

Cobb-Douglas production function was linearized so as to be able to estimate it using ordinary least square method. This is shown below:

The Cobb-Douglas production function is a special case of the CES function when $\sigma \rightarrow 1$ $that\ is\ when\ \rho \rightarrow 0$

The formal specification of the CES production function in its deterministic form is given by

$$Y = \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{v}{\rho}} \quad (1)$$

In Cobb-Douglas production function, $v = 1$ so that

$$Y = \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{1}{\rho}} \quad (2)$$

By finding the limits of the CES function as a function $Y(\rho)$ as $\rho \rightarrow 0$

Dividing both sides by γ and taking the natural log, we get an expression in the form

(3) $$ln\frac{Y}{\gamma} = ln[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{1}{\rho}}$$

(4) $$ln\frac{Y}{\gamma} = ln[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{1}{\rho}}$$

(5) $$lnG^{-\frac{a}{b}} = -\frac{a}{b}lnG$$ Recall that

(6) $$ln\frac{Y}{\gamma} = \left[\frac{-1}{\rho}ln[\delta K^{-\rho} + (1-\delta)L^{-\rho}]\right]$$

(7) $$ln\frac{Y}{\gamma} = -\frac{ln[\delta K^{-\rho}+(1-\delta)L^{-\rho}]}{\rho} \equiv \frac{m(\rho)}{n(\rho)}$$

As $\rho \rightarrow 0$, we find that $m(\rho) \rightarrow -ln[\delta + (1-\delta)] = -ln1 = 0$ and $m(\rho) \rightarrow 0$ as well. Applying L'Hopital's rule to find the limit of $ln\,(Y/\gamma)$. Once this done, the limit of Y can also be found since $(Y/\gamma) = e^{ln(Y/\gamma)}$so that $Y = \gamma e^{ln(Y/\gamma)}$, it follows that

$$limY = lim\gamma e^{ln(Y/\gamma)} = \gamma e^{limln(Y/\gamma)}$$

(8) $$\lim_{\rho\rightarrow 0} ln\frac{Y}{\gamma} = \lim_{\rho\rightarrow 0}\frac{m'(\rho)}{n\prime(\rho)}$$

(9) $$m'(\rho) = -\frac{\partial ln}{\partial \rho}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]$$

Recall that

$$\frac{\partial lna}{\partial b} = \frac{\partial a/\partial b}{a} = \frac{\partial a}{\partial b} \times \frac{1}{a} \quad (10)$$

Using chain rule,

(11) $$m'(\rho) = -\frac{\frac{\partial}{\partial\rho}[\delta K^{-\rho}+(1-\delta)L^{-\rho}]}{[\delta K^{-\rho}+(1-\delta)L^{-\rho}]}$$

(12) $$m'(\rho) = -\frac{1}{[\delta K^{-\rho}+(1-\delta)L^{-\rho}]} \times \frac{\partial}{\partial\rho}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]$$

(13) $$m'(\rho) = \frac{-1}{[\delta K^{-\rho}+(1-\delta)L^{-\rho}]} \times [-\delta K^{-\rho}lnK + (-(1-\delta)L^{-\rho}lnL)]$$

(14) $$m'(\rho) = \frac{-[-\delta K^{-\rho}lnK+(-(1-\delta)L^{-\rho}lnL)]}{[\delta K^{-\rho}+(1-\delta)L^{-\rho}]}$$

$$n'(\rho) = 1$$

We have

(15) $$\frac{m\prime(\rho)}{n\prime(\rho)} = \frac{\frac{-[-\delta K^{-\rho}lnK+(-(1-\delta)L^{-\rho}lnL)]}{[\delta K^{-\rho}+(1-\delta)L^{-\rho}]}}{1}$$

Therefore

(16) $$\lim_{\rho\to 0} ln\frac{Y}{\gamma} = \lim_{\rho\to 0}\frac{m'(\rho)}{n'(\rho)} = \frac{\frac{-[-\delta K^{-0}lnK+(-(1-\delta)L^{-0}lnL)]}{[\delta K^{-0}+(1-\delta)L^{-0}]}}{1}$$

(17) $$\lim_{\rho\to 0} ln\frac{Y}{\gamma} = \lim_{\rho\to 0}\frac{m'(\rho)}{n'(\rho)} = \frac{\frac{-[-\delta lnK+(-(1-\delta)lnL)]}{[\delta+(1-\delta)]}}{1}$$

(18) $$\lim_{\rho\to 0} ln\frac{Y}{\gamma} = \lim_{\rho\to 0}\frac{m'(\rho)}{n'(\rho)} = \frac{\frac{-[-\delta lnK(-(1-\delta)lnL)]}{[1]}}{1}$$

(19) $$\lim_{\rho\to 0} ln\frac{Y}{\gamma} = \lim_{\rho\to 0}\frac{m'(\rho)}{n'(\rho)} = -[-\delta lnK - (1-\delta)lnL] = [\delta lnK + (1-\delta)lnL]$$

(20) $$= ln\left(K^{\delta}L^{1-\delta}\right)$$

Recall that

(21) $$limY = lim\gamma e^{ln(Y/\gamma)} = \gamma e^{limln(Y/\gamma)}$$

(22) $$limY = \gamma e^{ln\left(K^{\delta}L^{1-\delta}\right)}$$

(23) $$\lim_{\rho\to 0} Y = \gamma K^{\delta}L^{1-\delta}$$

The deterministic form of celebrated Cobb-Douglas production maybe expressed as thus

(24) $$Y = \beta X_1^{\alpha_1}X_2^{\alpha_2}$$

Where $\gamma = \beta, \quad \delta = \alpha_1, \quad (1-\delta) = \alpha_1$

The stochastic form maybe expressed as

(25) $$Y = \beta X_1^{\alpha_1}X_2^{\alpha_2}e^{\mu_t}$$

In other to linearize this equation, the logarithm of both sides are taken

(26) $$lnY = \ln\left(\beta X_1^{\alpha_1}X_2^{\alpha_2}e^{\mu_t}\right)$$

Simplifying this equation, we have

(27) $$lnY = ln\beta + \alpha_1 lnX_1 + \alpha_2 lnX_2 + \mu_t$$

Where $ln\beta = \alpha_0$

(28) $$lnY = \alpha_0 + \alpha_1 lnX_1 + \alpha_2 lnX_2 + \mu_t$$

Where $Y = NGDP_t$, $X_1 = GFCF_t$ and $X_2 = LEMP_t$

So the linearized Cobb-Douglas production function for this study is

$$\boldsymbol{lnNGDP_t = \alpha_0 + \alpha_1 lnGFCF_t + \alpha_2 lnLEMP_t + \mu_t}$$ **(29)**

Constant Elasticity of Substitution Specification

The CES production function was also linearized to enable the use of ordinary square method in estimating.

The formal specification of the CES production function in its deterministic form is given by

$$Y = \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{v}{\rho}} \quad (30)$$

The stochastic form may be given by

$$Y = \gamma e^{\mu_t}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{v}{\rho}} \tag{31}$$

The CES function cannot be linearized analytically. However, Kmenta (1967) noted that the two-input CES function may be approximated by a Taylor series expansion when the parameter ρ is in the neighbourhood of zero (that is when the elasticity of substitution σ is in the neighbourhood of one).

Taylor's theorem, states that any arbitrary function $f\,(X)$ that is continuous and has a continuous nth-order derivative can be approximated around point $X = X_0$ by a polynomial function and a remainder as follows:

$$f(X) = \frac{f(X_0)}{0!} + \frac{f'(X_0)(X-X_0)}{1!} + \frac{f''(X_0)(X-X_0)^2}{2!} + \cdots + \frac{f^n(X_0)(X-X_0)^n}{n!} + R \tag{32}$$

Approximating the tradition CES function using the Taylor's theorem:

$$Y = \gamma e^{\mu_t}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{v}{\rho}} \tag{33}$$

Logarithmized CES function

$$lnY = \ln\gamma + \mu_t - \frac{v}{\rho}\ln(\delta K^{-\rho} + (1-\delta)L^{-\rho}) \tag{34}$$

Define function

$$f(\rho) \equiv -\frac{v}{\rho}\ln(\delta K^{-\rho} + (1-\delta)L^{-\rho}) \tag{35}$$

So that

$$lnY = ln\gamma + \mu_t + f(\rho) \tag{36}$$

Approximating the logarithm of the CES function by a first-order Taylor series when ρ is in the neighbourhood of zero ($\rho = 0$):

$$f(\rho) = \frac{f(0)}{0!} + \frac{f'(0)(\rho-0)}{1!} \tag{37}$$

$$lnY \approx ln\gamma + \mu_t + \frac{f(0)}{0!} + \frac{f'(0)(\rho-0)}{1!} \tag{38}$$

$$lnY \approx ln\gamma + \mu_t + f(0) + \rho f'(0) \tag{39}$$

The function is now define as

$$g(\rho) \equiv \delta K^{-\rho} + (1-\delta)L^{-\rho} \tag{40}$$

So that

$$f(\rho) = -\frac{v}{\rho}\ln\big(g(\rho)\big) \tag{41}$$

Calculating the first partial derivative of $f(\rho)$:

$$f'(\rho) = \frac{v}{\rho^2}\ln\big(g(\rho)\big) - \frac{v}{\rho}\partial\ln\,(g(\rho)) \tag{42}$$

$$f'(\rho) = \frac{v}{\rho^2}\ln\big(g(\rho)\big) - \frac{v}{\rho}\,\frac{g'(\rho)}{g(\rho)} \tag{43}$$

Recall that $\frac{d(a^x)}{dx} = a^x loga, \quad a\ is$ a constant

The three first derivatives of $g(\rho)$ are given below

$$g'(\rho) = -\delta K^{-\rho}lnK - (1-\delta)L^{-\rho}lnL \tag{44}$$

$$g''(\rho) = \delta K^{-\rho}(lnK)^2 + (1-\delta)L^{-\rho}(lnL)^2 \tag{45}$$

$$g''' = -\delta K^{-\rho}(lnK)^3 - (1-\delta)L^{-\rho}(lnL)^3 \tag{46}$$

At the point of approximation $\rho = 0$, we have

$$g(0) = \delta K^{-0} + (1-\delta)L^{-0} \tag{47}$$

$$g(0) = \delta + 1 - \delta \tag{48}$$

$$g(0) = 1 \tag{49}$$

$$g'(0) = -\delta K^{-0}lnK - (1-\delta)L^{-0}lnL \tag{50}$$

$$g'(0) = -\delta lnK - (1-\delta)lnL \tag{51}$$

$$g''(0) = \delta K^{-0}(lnK)^2 + (1-\delta)L^{-0}(lnL)^2 \tag{52}$$

$$g''(0) = \delta(lnK)^2 + (1-\delta)(lnL)^2 \tag{53}$$

$$g'''(0) = -\delta K^{-0}(lnK)^3 - (1-\delta)L^{-0}(lnL)^3 \quad (54)$$
$$g'''(0) = -\delta(lnK)^3 - (1-\delta)(lnL)^3 \quad (55)$$

Calculating the limit of $f(\rho)$ for $\rho \to 0$:

$$f(0) = \lim_{\rho\to 0} f(\rho) \quad (56)$$

$$= \lim_{\rho\to 0} \frac{-vln(g(\rho))}{\rho} \quad (57)$$

Recall L'Hopital's Rule II: if $f(x_0) =< g(x_0) = \infty, then$:

$$\lim_{x\to x_o} \frac{f(x)}{g(x)} = \lim_{x\to x_o} \frac{f'(x)}{g'(x)} \quad (58)$$
$$= \lim_{\rho\to 0} \frac{\partial(-vln(g(\rho)))}{\partial(\rho)} \quad (59)$$
$$= \lim_{\rho\to 0} \frac{-v\frac{g'(\rho)}{g(\rho)}}{1} \quad (60)$$
$$= \frac{-v[-\delta K^{-\rho} lnK - (1-\delta)L^{-\rho} lnL]}{\delta K^{-\rho} + (1-\delta)L^{-\rho}} \quad (61)$$
$$= \frac{-v[-\delta K^{-0} lnK - (1-\delta)L^{-0} lnL]}{\delta K^{-0} + (1-\delta)L^{-0}} \quad (62)$$
$$= \frac{-v[-\delta lnK - (1-\delta)lnL]}{\delta + (1-\delta)} \quad (63)$$
$$= \frac{v[\delta lnK + (1-\delta)lnL]}{1} \quad (64)$$
$$f(0) = v[\delta lnK + (1-\delta)lnL] \quad (65)$$

And the limits of $f'(\rho) \to 0$:

$$f'(0) = \lim_{\rho\to 0} f'(\rho) \quad (66)$$
$$= \lim_{\rho\to 0} \left(\frac{v}{\rho^2} lng(\rho)) - \frac{v}{\rho}\frac{g'(\rho)}{g(p)}\right) \quad (67)$$
$$= \lim_{\rho\to 0} \left(\frac{vlng(\rho) - v\rho\frac{g'(\rho)}{g(\rho)}}{\rho^2}\right) \quad (68)$$

Applying L'Hopital's Rule II

$$= \lim_{\rho\to 0} \frac{\partial\left(vlng(\rho) - v\rho\frac{g'(\rho)}{g(\rho)}\right)}{\partial(\rho^2)} \quad (69)$$
$$= \lim_{\rho\to 0} \left(\frac{v\frac{g'(\rho)}{g(\rho)} - v\frac{g'(\rho)}{g(\rho)} - v\rho\frac{g(\rho)g''(\rho) - (g'(\rho))^2}{g(\rho)^2}}{2\rho}\right) \quad (70)$$
$$= \lim_{\rho\to 0} \frac{-v\rho\frac{g(\rho)g''(\rho) - (g'(\rho))^2}{g(\rho)^2}}{2\rho} \quad (71)$$
$$= \lim_{\rho\to 0} \frac{-v\,\frac{g(\rho)g''(\rho) - (g'(\rho))^2}{g(\rho)^2}}{2} \quad (72)$$
$$= \lim_{\rho\to 0} -\frac{v}{2}\;\frac{g(\rho)g''(\rho) - (g'(\rho))^2}{g(\rho)^2} \quad (73)$$
$$= -\frac{v}{2}\;\frac{g(0)g''(0) - (g'(0))^2}{g(0)^2} \quad (74)$$
$$= -\frac{v}{2}\;\frac{(1)\left(\delta(lnK)^2 + (1-\delta)(lnL)^2\right) - (-\delta lnK - (1-\delta)lnL)^2}{(1)^2} \quad (75)$$
$$= -\frac{v}{2}\;\left((\delta(lnK)^2 + (1-\delta)(lnL)^2) - ((-\delta lnK - (1-\delta)lnL)^2)\right) \quad (76)$$
$$= -\frac{v}{2}\;(\delta(lnK)^2 + (1-\delta)(lnL)^2) - (-\delta lnK - lnL + \delta lnL)^2 \quad (77)$$

$= -\frac{v}{2}(\delta(lnK)^2 + (1-\delta)(lnL)^2 - (\delta^2(lnK)^2 + 2\delta lnKlnL - 2\delta^2 lnKlnL + (lnL)^2 - 2\delta(lnL)^2 + \delta^2(lnL)^2))$ (78)

$= -\frac{v}{2}(\delta(lnK)^2 + (1-\delta)(lnL)^2 - (+(2-2\delta^2)lnKlnL + (1-2\delta+\delta^2)(lnL)^2)$ (79)*

$= -\frac{v}{2}(\delta(lnK)^2 + (1-\delta)(lnL)^2 - (\delta^2(lnK)^2 + 2\delta(1-\delta)lnKlnL + (1-\delta)^2(lnL)^2)$ (80)

$= -\frac{v}{2}(\delta(lnK)^2 + (1-\delta)(lnL)^2 - \delta^2(lnK)^2 - 2\delta(1-\delta)lnKlnL - (1-\delta)^2(lnL)^2$ (81)

Collect like terms

$= -\frac{v}{2}(\delta(lnK)^2 - \delta^2(lnK)^2 + (1-\delta)(lnL)^2 - (1-\delta)^2(lnL)^2 - 2\delta(1-\delta)lnKlnL$ (82)

$= -\frac{v}{2}(\delta(1-\delta)(lnK)^2 + (1-\delta)(1-(1-\delta))(lnL)^2 - 2\delta(1-\delta)lnKlnL$ (83)

$= -\frac{v}{2}(\delta(1-\delta)(lnK)^2 + \delta(1-\delta)(lnL)^2 - 2\delta(1-\delta)lnKlnL$ (84)

Factorize

$$= -\frac{v\delta(1-\delta)}{2}((lnK)^2 - 2lnKlnL + (lnL)^2) \quad (85)$$

$$f'(0) = -\frac{v\delta(1-\delta)}{2}(lnK - lnL)^2 \quad (86)$$

So that we get the following first-order Taylor series approximation around ρ=0

$= lnY \approx \ln\gamma + \mu_t + v[\delta lnK + (1-\delta)lnL] - \frac{\rho v\delta(1-\delta)}{2}[(lnK - lnL)^2]$ (87)

$= lnY \approx \ln\gamma + \mu_t + v\delta lnK + v(1-\delta)lnL - \frac{\rho v\delta(1-\delta)}{2}[(lnK - lnL)^2]$ (88)

$= lnY \approx \ln\gamma + v\delta lnK + v(1-\delta)lnL - \frac{\rho v\delta(1-\delta)}{2}[(lnK - lnL)^2] + \mu_t$ (89)

The function given in equation (58) is recognized as a translog function

$lnY = \alpha_0 + \alpha_1(lnK) + \alpha_2(lnL) + \alpha_3(lnK - lnL)^2$ (90)

The parameters fulfill the conditions:

$\alpha_0 = ln\gamma \quad \alpha_1 = v\delta \quad \alpha_2 = v(1-\delta) = v - v\delta = v - \alpha_1 \quad \alpha_1 + \alpha_2 = v$

$\alpha_3 = \frac{\rho v\delta(1-\delta)}{2}$

Since for this study,

$$Y = NGDP, \quad K = GFCF, \quad L = LEMP$$

Our translog constant elasticity of substitution for this study is

(91) $lnNGDP_t = \alpha_0 + \alpha_1 lnGFCF_t + \alpha_2 lnLEMP_t + \alpha_3(lnGFCF_t - lnLEMP_t)^2 + \mu_t$

APPENDIX A2

ESTIMATION OF THE COBB-DOUGLAS AND CONSTANT ELASTICITY OF SUBSTITUTION SPECIFICATION

Cobb-Douglas Specification

$$NGDP_t = \beta GFCF_t^{\alpha_1} LEMP_t^{\alpha_2} \quad (92)$$

And $\boldsymbol{\alpha_0 = ln\beta}$

From model 1a, the representation of the linearized Cobb-Douglas specification is given as

$$\boldsymbol{lnNGDP_t = \alpha_0 + \alpha_1 lnGFCF_t + \alpha_2 lnLEMP_t}$$

(93)

$$\boldsymbol{lnNGDP_t = -54.1186 + 0.6812225 lnGFCF_t + 3.476717 lnLEMP_t}$$

(94)

To find β, the antilogarithm of $\boldsymbol{\alpha_0}$ will be taken

$$\beta = e^{\alpha_0} = e^{-54.1186} = 3.137549973 \times 10^{-24} \quad (95)$$

The unlinearized Cobb-Douglas function can be written as

$$NGDP_t = 3.137549973 \times 10^{-24} GFCF_t^{\ 0.6812225} LEMP_t^{\ 3.476717} \quad (96)$$

Constant Elasticity of Substitution Specification

The unlinearized Constant elasticity of substitution specification is given as

$$NGDP_t = \gamma[\delta GFCF_t^{\ -\rho} + (1-\delta)LEMP_t^{\ -\rho}]^{-\frac{v}{\rho}} \quad (97)$$

When linearized it becomes

$$lnNGDP_t = \ln\gamma + \ v\delta lnGFCF_t + v(1-\delta)lnLEMP_t - \frac{\rho v\delta(1-\delta)}{2}[(lnLEMP_t - lnGFCF_t)^2] \quad (98)$$

And then this

$$\boldsymbol{lnNGDP_t = \alpha_0 + \ \alpha_1 lnGFCF_t + \alpha_2 lnLEMP_t + \alpha_3(lnGFCF_t - lnLEMP_t)^2} \quad \mathbf{(99)}$$

When model 1b was estimated it became

$$lnNGDP_t = -51.65402 + \ 0.5602974 lnGFCF_t + 3.442377 lnLEMP_t + (-0.0133799)(lnGFCF_t - lnLEMP_t)^2 \quad (100)$$

Now

$$\gamma = e^{\alpha_0} = e^{-51.65402} = 3.689301478 \times 10^{-23} \quad (101)$$

$$v\delta = \alpha_1 = 0.5602974 \quad (102)$$

$$v(1-\delta) = \alpha_2 = \ 3.442377 \quad (103)$$

$$\frac{\rho v\delta(1-\delta)}{2} = \alpha_3 = -0.0133799 \quad (104)$$

From (102)

$$v = \frac{0.5602974}{\delta} \quad (105)$$

Form (103)

$$v = \frac{3.442377}{(1-\delta)} \quad (106)$$

Equating (105) and (106) we have

$$v = \frac{0.5602974}{\delta} = \frac{3.442377}{(1-\delta)} \quad (107)$$

Cross multiply

$$3.442377\delta = 0.5602974(1-\delta) \quad (108)$$

Expanding,

$$3.442377\delta = 0.5602974 - 0.5602974\delta \quad (109)$$

Collect like terms

$$3.4442377\delta + 0.5602974\delta = 0.5602974 \quad (110)$$

$$4.00267744\delta = 0.5602974 \quad (111)$$

$$\delta = \frac{0.5602974}{4.0026744} = 0.139980758864623 \quad (112)$$

From (105)

$$v = \frac{0.5602974}{\delta} = \frac{0.5602974}{0.139980758864623} = 4.0026744 \quad (113)$$

From (106)

$$v = \frac{3.442377}{(1-\delta)} = \frac{3.442377}{(1-0.139980758864623)} = \frac{3.442377}{(0.860019241135377)} = 4.0026744 \quad (114)$$

From (104)

$$\frac{\rho v\delta(1-\delta)}{2} = -0.0133799$$

Cross multiply

$$\rho v\delta(1-\delta) = -0.0267598 \quad (115)$$

Divide both sides by $v\delta(1-\delta)$

$$\rho = \frac{-0.0267598}{v\delta(1-\delta)} \quad (116)$$

Substitute the values of v and δ

$$\rho = \frac{-0.0267598}{v\delta(1-\delta)} = \frac{-0.0267598}{4.0026744 \times 0.139980758864623(1-0.139980758864623)} = \frac{-0.0267598}{0.481866544758124} = -0.0555336333080194 \quad (117)$$

$$\gamma = 3.689301478 \times 10^{-23} \quad (118)$$

$$\delta = 0.139980758864623 \quad (119)$$

$$v = 4.0026744 \quad (120)$$

$$\rho = -0.0555336333080194 \quad (121)$$

$$NGDP_t = 3.689301478 \times 10^{-23}\left[0.139980758864623 GFCF_t^{0.0555336333080194} + (1 - 0.139980758864623) LEMP_t^{0.0555336333080194}\right]^{\frac{4.0026744}{0.0555336333080194}} \quad (122)$$

The unlinearized Constant elasticity of substitution production function can be written as

$$NGDP_t = 3.689301478 \times 10^{-23}\left[0.139980758864623 GFCF_t^{0.0555336333080194} + 0.860019241135377 LEMP_t^{0.0555336333080194}\right]^{7.20765806515669} \quad (123)$$